Дисперсия линейной комбинации

$$\mathbf{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \mathrm{E}\left(\left(\sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i \mu_i\right)^2\right) = \mathrm{E}\left(\left(\sum_{i=1}^{n} a_i (X_i - \mu_i)\right)^2\right) =$$

$$= \mathrm{E}\left(\sum_{i=1}^{n} \left(a_i (X_i - \mu_i)\right)^2 + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i (X_i - \mu_i) \cdot a_j \left(X_j - \mu_j\right)\right) =$$

$$= \sum_{i=1}^{n} a_i^2 \mathrm{E}((X_i - \mu_i)^2) + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i a_j \mathrm{E}\left((X_i - \mu_i)\left(X_j - \mu_j\right)\right)$$

$$\mathbf{Var}\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i^2 \mathbf{Var}(X_i) + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_i a_j \mathbf{Cov}\left(X_i; X_j\right)$$

Дисперсия среднего выборочного

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

$$\mathbf{Cov}\left(X_i; X_j\right) = \delta_{ij}\sigma_X^2 \equiv \begin{cases} \sigma_X^2 & i = j \\ 0 & i \neq j \end{cases}$$

$$\mathrm{Var}(X_i) = \mathrm{Var}(X) = \sigma_X^2$$

$$2\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Cov}\left(X_i; X_j\right) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \mathrm{Cov}\left(X_i; X_j\right) = 0$$

$$\mathbf{Var}\left(\overline{X}\right) = \mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) + \frac{2}{n^2}\sum_{i=1}^{n} \sum_{j=i+1}^{n} \mathrm{Cov}\left(X_i; X_j\right)$$

$$\mathbf{Var}\left(\overline{X}\right) = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}(X_i) = \frac{1}{n^2} n \mathrm{Var}(X) = \frac{\sigma_X^2}{n}$$

Несмещенная оценка выборочной дисперсии

$$\mathbf{E}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\right)=\frac{1}{n-1}\mathrm{E}\left(\sum_{i=1}^{n}\left((X_i-\mu_X)-(\overline{X}-\mu_X)\right)^2\right)=$$

$$=\frac{1}{n-1}\mathrm{E}\left(\sum_{i=1}^{n}(X_i-\mu_X)^2+\sum_{i=1}^{n}(\overline{X}-\mu_X)^2-2\sum_{i=1}^{n}(X_i-\mu_X)(\overline{X}-\mu_X)\right)=$$

$$=\frac{1}{n-1}\mathrm{E}\left(\sum_{i=1}^{n}(X_i-\mu_X)^2+n(\overline{X}-\mu_X)^2-2\sum_{i=1}^{n}(X_i-\mu_X)(\overline{X}-\mu_X)\right)=$$

$$=\frac{1}{n-1}\left(\sum_{i=1}^{n}\mathbf{E}((X_i-\mu_X)^2)+n\mathbf{E}\left((\overline{X}-\mu_X)^2\right)-2\sum_{i=1}^{n}\mathbf{E}\left((X_i-\mu_X)(\overline{X}-\mu_X)\right)\right)$$

$$\overline{X}=\frac{1}{n}\sum_{i=1}^{n}X_i \qquad \mathrm{Cov}(X_i;X_j)=\delta_{ij}\sigma_X^2\equiv\begin{cases}\sigma_X^2 & i=j\\ 0 & i\neq j\end{cases}$$

$$\mathbf{E}((X_i-\mu_X)^2)=\mathrm{Var}(X_i)=\mathrm{Var}(X)=\sigma_X^2$$

$$\mathbf{E}\left((\overline{X}-\mu_X)^2\right)=\mathrm{Var}(\overline{X})=\frac{\sigma_X^2}{n}$$

$$\mathbf{E}\left((X_i-\mu_X)(\overline{X}-\mu_X)\right)=\mathrm{Cov}(X_i;\overline{X})=\mathrm{Cov}\left(X_i;\frac{1}{n}\sum_{j=1}^{n}X_j\right)=\frac{1}{n}\sum_{j=1}^{n}\mathrm{Cov}(X_i;X_j)=\frac{\sigma_X^2}{n}$$

$$\mathbf{E}\left(\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\overline{X})^2\right)=\frac{1}{n-1}\left(n\sigma_X^2+n\frac{\sigma_X^2}{n}-2n\frac{\sigma_X^2}{n}\right)=\sigma_X^2$$

М е т о д   Н а и м е н ь ш и х   К в а д р а т о в

$$Y = \mathbb{X}\beta + U \qquad\qquad U = Y - \mathbb{X}\beta$$

$$\boldsymbol{RSS} = U^T U = (Y - \mathbb{X}\beta)^T (Y - \mathbb{X}\beta) = \left(Y^T - \beta^T \mathbb{X}^T\right)(Y - \mathbb{X}\beta) =$$

$$= Y^T Y - \boldsymbol{Y^T \mathbb{X} \beta} - \boldsymbol{\beta^T \mathbb{X}^T Y} + \boldsymbol{\beta^T \mathbb{X}^T \mathbb{X} \beta}$$

$$Y^T \mathbb{X}\beta = \boxed{1 \times 1} \quad \Rightarrow \quad Y^T \mathbb{X}\beta = \left(Y^T \mathbb{X}\beta\right)^T = \beta^T \mathbb{X}^T Y$$

$$\frac{\partial(\dots)}{\partial \beta^T} = \left(\frac{\partial(\dots)}{\partial \beta}\right)^T$$

$$\frac{\partial\left(\boldsymbol{Y^T \mathbb{X} \beta}\right)}{\partial \beta^T} = \left(\frac{\partial\left(Y^T \mathbb{X}\beta\right)}{\partial \beta}\right)^T = \left(Y^T \mathbb{X}\right)^T = \boldsymbol{\mathbb{X}^T Y}$$

$$\frac{\partial\left(\boldsymbol{\beta^T \mathbb{X}^T Y}\right)}{\partial \beta^T} = \boldsymbol{\mathbb{X}^T Y}$$

$$\frac{\partial\left(\boldsymbol{\beta^T \mathbb{X}^T \mathbb{X} \beta}\right)}{\partial \beta^T} = \frac{\partial\left(\overset{\downarrow}{\widetilde{\beta^T}} \mathbb{X}^T \mathbb{X}\beta\right)}{\partial \beta^T} + \frac{\partial\left(\beta^T \mathbb{X}^T \mathbb{X} \overset{\downarrow}{\tilde{\beta}}\right)}{\partial \beta^T} = \mathbb{X}^T \mathbb{X}\beta + \left(\frac{\partial\left(\beta^T \mathbb{X}^T \mathbb{X} \overset{\downarrow}{\tilde{\beta}}\right)}{\partial \beta}\right)^T$$

$$= \mathbb{X}^T \mathbb{X}\beta + \left(\beta^T \mathbb{X}^T \mathbb{X}\right)^T = \boldsymbol{2\mathbb{X}^T \mathbb{X}\beta}$$

$$\frac{\partial RSS}{\partial \beta^T} = -\boldsymbol{\mathbb{X}^T Y} - \boldsymbol{\mathbb{X}^T Y} + \boldsymbol{2\mathbb{X}^T \mathbb{X}\beta} = -2\mathbb{X}^T (Y - \mathbb{X}\beta) = -2\mathbb{X}^T U = 0$$

$$\boldsymbol{\mathbb{X}^T U = 0} \qquad\qquad \mathbb{X}^T Y = \mathbb{X}^T \mathbb{X}\beta$$

$$\boldsymbol{\beta = \left(\mathbb{X}^T \mathbb{X}\right)^{-1} \mathbb{X}^T Y}$$

$$\beta = \begin{pmatrix} 1 & \overline{X} \\ \overline{X} & \overline{X^2} \end{pmatrix}^{-1} \begin{pmatrix} \overline{Y} \\ \overline{XY} \end{pmatrix} = \frac{1}{s_X^2} \begin{pmatrix} \overline{X^2} & -\overline{X} \\ -\overline{X} & 1 \end{pmatrix} \begin{pmatrix} \overline{Y} \\ \overline{XY} \end{pmatrix} = \frac{1}{s_X^2} \begin{pmatrix} \overline{Y} \cdot \overline{X^2} - \overline{X} \cdot \overline{XY} \\ \overline{XY} - \overline{X} \cdot \overline{Y} \end{pmatrix} =$$

$$= \frac{1}{s_X^2} \begin{pmatrix} \overline{Y} \cdot \overline{X^2} - \overline{Y} \cdot \overline{X}^2 + \overline{Y} \cdot \overline{X}^2 - \overline{X} \cdot \overline{XY} \\ \overline{XY} - \overline{X} \cdot \overline{Y} \end{pmatrix} = \frac{1}{s_X^2} \begin{pmatrix} \overline{Y} s_X^2 - \overline{X}\left(\overline{XY} - \overline{Y} \cdot \overline{X}\right) \\ \overline{XY} - \overline{X} \cdot \overline{Y} \end{pmatrix} = \begin{pmatrix} \overline{Y} - \overline{X}\dfrac{\sigma_{XY}}{\sigma_X^2} \\ \dfrac{\sigma_{XY}}{\sigma_X^2} \end{pmatrix}$$

Т е о р е м а   Г а у с с а – М а р к о в а

$$y_i = b_0 + b_1 x_i + \varepsilon_i$$

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

$$\mathbf{E}(\boldsymbol{\varepsilon_i}) = \mathbf{0}$$

$$\mathbf{Var}(\boldsymbol{\varepsilon_i}) = \boldsymbol{\sigma_\varepsilon^2} = \mathbf{const}$$

$$\mathbf{Cov}(\boldsymbol{\varepsilon_i}; \boldsymbol{\varepsilon_j}) = \boldsymbol{\delta_{ij}\sigma_\varepsilon^2} \equiv \begin{cases} \boldsymbol{\sigma_\varepsilon^2} & \boldsymbol{i = j} \\ \mathbf{0} & \boldsymbol{i \neq j} \end{cases}$$

$$\boldsymbol{X} = \mathbf{const}$$

$$w_i = \frac{x_i - \overline{X}}{\sigma_x^2 (n-1)} \qquad \sum_{i=1}^{n} w_i = 0$$

$$\sum_{i=1}^{n} w_i x_i = \sum_{i=1}^{n} w_i (x_i - \overline{X}) = \sum_{i=1}^{n} \frac{x_i - \overline{X}}{\sigma_x^2 (n-1)} (x_i - \overline{X}) = \frac{1}{\sigma_x^2 (n-1)} \sum_{i=1}^{n} (x_i - \overline{X})^2 = 1$$

$$\beta_1 = \frac{1}{\sigma_x^2 (n-1)} \sum_{i=1}^{n} (x_i - \overline{X})(Y_i - \overline{Y}) = \sum_{i=1}^{n} w_i Y_i = \sum_{i=1}^{n} w_i (b_0 + b_1 x_i + \varepsilon_i)$$

$$\beta_1 = b_0 \sum_{i=1}^{n} w_i + b_1 \sum_{i=1}^{n} w_i x_i + \sum_{i=1}^{n} w_i \varepsilon_i = b_1 + \sum_{i=1}^{n} w_i \varepsilon_i$$

<u>Н е с м е щ е н н о с т ь</u>

$$\mathbf{E}(\boldsymbol{\beta_1}) = \mathrm{E}\left(b_1 + \sum_{i=1}^{n} w_i \varepsilon_i\right) = b_1 + \sum_{i=1}^{n} w_i \mathrm{E}(\varepsilon_i) = \boldsymbol{b_1}$$

$$\beta_0 = \overline{Y} - \beta_1 \overline{X}$$

$$\overline{Y} = b_0 + b_1 \overline{X} + \overline{\varepsilon}$$

$$\beta_0 = (b_0 + b_1 \overline{X} + \overline{\varepsilon}) - \beta_1 \overline{X} = b_0 + (b_1 - \beta_1)\overline{X} + \overline{\varepsilon}$$

$$\mathbf{E}(\boldsymbol{\beta_0}) = b_0 + (b_1 - \mathrm{E}(\beta_1))\overline{X} = \boldsymbol{b_0}$$

Эффективность

$$\mathrm{Var}(\beta_1) = \mathrm{Var}\left(b_1 + \sum_{i=1}^{n} w_i\varepsilon_i\right) = \sum_{i=1}^{n} w_i^2\mathrm{Var}(\varepsilon_i) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} w_i w_j \mathrm{Cov}(\varepsilon_i;\varepsilon_j)$$

$$\sum_{i=1}^{n} w_i^2 = \sum_{i=1}^{n} w_i \frac{x_i - \overline{X}}{\sigma_x^2(n-1)} = \frac{1}{\sigma_x^2(n-1)}\sum_{i=1}^{n} w_i x_i = \frac{1}{\sigma_x^2(n-1)} \qquad \boldsymbol{\mathrm{Var}(\beta_1) = \frac{\sigma_\varepsilon^2}{(n-1)\sigma_x^2}}$$

$$\tilde{\beta}_1 = \sum_{i=1}^{n} \tilde{w}_i Y_i = \sum_{i=1}^{n}(w_i + h_i)Y_i = \beta_1 + \sum_{i=1}^{n} h_i Y_i$$

$$\mathrm{E}(\tilde{\beta}_1) = \mathrm{E}(\beta_1) + \sum_{i=1}^{n} h_i \mathrm{E}(b_0 + b_1 x_i + \varepsilon_i)$$

$$\mathrm{E}(\tilde{\beta}_1) = b_1 + b_0\sum_{i=1}^{n} h_i + b_1\sum_{i=1}^{n} h_i x_i + \sum_{i=1}^{n} h_i \mathrm{E}(\varepsilon_i) = b_1 + b_0\sum_{i=1}^{n} h_i + b_1\sum_{i=1}^{n} h_i x_i$$

$$\mathrm{E}(\tilde{\beta}_1) = b_1 \quad \Longrightarrow \quad b_0\sum_{i=1}^{n} h_i + b_1\sum_{i=1}^{n} h_i x_i = 0 \quad \Longrightarrow$$

$$\boldsymbol{\sum_{i=1}^{n} h_i = 0} \qquad \boldsymbol{\sum_{i=1}^{n} h_i x_i = 0}$$

$$\tilde{\beta}_1 = \beta_1 + \sum_{i=1}^{n} h_i(b_0 + b_1 x_i + \varepsilon_i) = \beta_1 + \sum_{i=1}^{n} h_i\varepsilon_i$$

$$\mathrm{Var}(\tilde{\beta}_1) = \mathrm{Var}(\beta_1) + \sum_{i=1}^{n} h_i^2\mathrm{Var}(\varepsilon_i) + 2\sum_{i=1}^{n} h_i \boldsymbol{\mathrm{Cov}(\beta_1;\varepsilon_i)} + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} h_i h_j \boldsymbol{\mathrm{Cov}(\varepsilon_i;\varepsilon_j)}$$

$$\boldsymbol{\mathrm{Cov}(\beta_1;\varepsilon_i)} = \mathrm{Cov}\left(\left(b_1 + \sum_{j=1}^{n} w_j\varepsilon_j\right);\varepsilon_i\right) = \sum_{j=1}^{n} w_j \mathrm{Cov}(\varepsilon_j;\varepsilon_i) = w_i\sigma_\varepsilon^2$$

$$\sum_{i=1}^{n} h_i \boldsymbol{\mathrm{Cov}(\beta_1;\varepsilon_i)} = \sigma_\varepsilon^2\sum_{i=1}^{n} h_i w_i = \sigma_\varepsilon^2\sum_{i=1}^{n} h_i\frac{x_i - \overline{X}}{\sigma_x^2(n-1)} = \frac{\sigma_\varepsilon^2}{\sigma_x^2(n-1)}\left(\sum_{i=1}^{n} h_i x_i - \overline{X}\sum_{i=1}^{n} h_i\right) = 0$$

$$\boldsymbol{\mathrm{Var}(\tilde{\beta}_1) = \mathrm{Var}(\beta_1) + \sigma_\varepsilon^2\sum_{i=1}^{n} h_i^2 > \mathrm{Var}(\beta_1)}$$

О ц е н к а   д и с п е р с и и   о ш и б о к

$$Y_i = b_0 + b_1 X_i^1 + b_2 X_i^2 + \dots + b_{(k-1)} X_i^{(k-1)} + \varepsilon_i = b_0 + \sum_{l=1}^{k-1} b_l X_i^l + \varepsilon_i$$

$$Y_i = \beta_0 + \beta_1 X_i^1 + \beta_2 X_i^2 + \dots + \beta_{(k-1)} X_i^{(k-1)} + U_i = \beta_0 + \sum_{l=1}^{k-1} \beta_l X_i^l + U_i$$

$$b_0 = \overline{Y} - \sum_{l=1}^{k-1} b_l \overline{X^l} - \overline{\varepsilon} \qquad \beta_0 = \overline{Y} - \sum_{l=1}^{k-1} \beta_l \overline{X^l}$$

$$U_i = \overbrace{\left(b_0 + \sum_{l=1}^{k-1} b_l X_i^l + \varepsilon\right)}^{Y} - \overbrace{\left(\overline{Y} - \sum_{l=1}^{k-1} \beta_l \overline{X^l}\right)}^{\beta_0} - \sum_{l=1}^{k-1} \beta_l X_i^l$$

$$U_i = \overbrace{\left(\overline{\boldsymbol{Y}} - \sum_{l=1}^{k-1} b_l \overline{X^l} - \overline{\varepsilon}\right)}^{b_0} + \sum_{l=1}^{k-1} b_l X_i^l + \varepsilon_i - \left(\overline{\boldsymbol{Y}} - \sum_{l=1}^{k-1} \beta_l \overline{X^l}\right) - \sum_{l=1}^{k-1} \beta_l X_i^l$$

$$\boldsymbol{U_i = \sum_{l=1}^{k-1} (b_l - \beta_l)\left(X_i^l - \overline{X^l}\right) + \varepsilon_i - \overline{\varepsilon}}$$

$\boldsymbol{X = \mathrm{const}}$

$$\mathrm{E}(U_i) = \sum_{l=1}^{k-1} \mathrm{E}(b_l - \beta_l) \cdot \left(X_i^l - \overline{X^l}\right) + \mathrm{E}(\varepsilon_i - \overline{\varepsilon}) = 0$$

$$\mathrm{Var}(U_i) = \mathrm{E}\left(U_i^2\right) - \mathrm{E}^2(U_i) = \mathrm{E}\left(U_i^2\right)$$

$$RSS = \sum_{i=1}^{n} U_i^2 \qquad \mathrm{E}(RSS) = \sum_{i=1}^{n} \mathrm{E}\left(U_i^2\right) = \sum_{i=1}^{n} \mathrm{Var}(U_i)$$

$$\boldsymbol{\mathrm{Var}(U_i)} = \sum_{l=1}^{k-1} \left(x_i^l - \overline{X^l}\right)^2 \mathrm{Var}(\beta_l) + \mathrm{Var}(\varepsilon_i) + \mathrm{Var}(\overline{\varepsilon}) - 2\sum_{l=1}^{k-1} \left(x_i^l - \overline{X^l}\right)\left(\boldsymbol{\mathrm{Cov}(\beta_l;\varepsilon_i)} + \boldsymbol{\mathrm{Cov}(\beta_l;\overline{\varepsilon})}\right) - 2\boldsymbol{\mathrm{Cov}(\varepsilon_i;\overline{\varepsilon})}$$

$$\mathrm{Var}(\varepsilon_i) = \sigma_\varepsilon^2 \qquad \mathrm{Var}(\overline{\varepsilon}) = \frac{\sigma_\varepsilon^2}{n} \qquad \mathrm{Var}(\beta_l) = \frac{\sigma_\varepsilon^2}{(n-1)\cdot \mathrm{Var}(X^l)}$$

$$\mathrm{Cov}(\beta_l;\beta_m) = 0 \qquad l \neq m$$

$$\mathrm{Cov}\left(\varepsilon_i;\varepsilon_j\right) = 0 \qquad i \neq j$$

$$\mathrm{Cov}(X^l;X^m) = 0 \qquad l \neq m$$

$$\mathbf{Cov}(\boldsymbol{\varepsilon_i};\overline{\boldsymbol{\varepsilon}}) = \frac{1}{n}\sum_{j=1}^{n}\mathrm{Cov}\left(\varepsilon_i;\varepsilon_j\right) = \frac{\boldsymbol{\sigma_\varepsilon^2}}{\boldsymbol{n}}$$

$$\beta_l = \frac{1}{\sigma_l^2(n-1)}\sum_{j=1}^{n} Y_j\left(x_j^l - \overline{X^l}\right) = \frac{1}{\sigma_l^2(n-1)}\sum_{j=1}^{n}\left(b_0 + \sum_{m=1}^{k-1} b_m x_j^m + \boldsymbol{\varepsilon_j}\right)\left(\boldsymbol{x_j^l} - \overline{\boldsymbol{X^l}}\right)$$

$$\mathbf{Cov}(\boldsymbol{\beta_l};\boldsymbol{\varepsilon_i}) = \frac{1}{\sigma_l^2(n-1)}\sum_{j=1}^{n}\left(x_j^l - \overline{X^l}\right)\mathrm{Cov}\left(\varepsilon_j;\varepsilon_i\right) = \frac{\boldsymbol{\sigma_\varepsilon^2}}{\boldsymbol{\sigma_l^2(n-1)}}\left(\boldsymbol{x_i^l} - \overline{\boldsymbol{X^l}}\right)$$

$$\mathbf{Cov}(\boldsymbol{\beta_l};\overline{\boldsymbol{\varepsilon}}) = \frac{1}{n}\sum_{i=1}^{n}\mathbf{Cov}(\boldsymbol{\beta_l};\boldsymbol{\varepsilon_i}) = \frac{1}{n}\sum_{i=1}^{n}\frac{\sigma_\varepsilon^2}{\sigma_l^2(n-1)}\left(x_j^l - \overline{X^l}\right) = \mathbf{0}$$

$$\mathbf{Var}(\boldsymbol{U_i}) = \sum_{l=1}^{k-1}\left(x_i^l - \overline{X^l}\right)^2\frac{\sigma_\varepsilon^2}{\sigma_l^2(n-1)} + \sigma_\varepsilon^2 + \frac{\sigma_\varepsilon^2}{n} - 2\sum_{l=1}^{k-1}\left(x_i^l - \overline{X^l}\right)\frac{\boldsymbol{\sigma_\varepsilon^2}}{\boldsymbol{\sigma_l^2(n-1)}}\left(\boldsymbol{x_i^l} - \overline{\boldsymbol{X^l}}\right) - 2\frac{\boldsymbol{\sigma_\varepsilon^2}}{\boldsymbol{n}}$$

$$\mathrm{Var}(U_i) = -\frac{\sigma_\varepsilon^2}{(n-1)}\sum_{l=1}^{k-1}\frac{1}{\sigma_l^2}\left(x_i^l - \overline{X^l}\right)^2 + \sigma_\varepsilon^2 - \frac{\sigma_\varepsilon^2}{n}$$

$$\frac{1}{(n-1)\sigma_l^2}\sum_{i=1}^{n}\left(x_i^l - \overline{X^l}\right)^2 = 1$$

$$\mathrm{E}(RSS) = \sum_{i=1}^{n}\mathrm{Var}(U_i) = \sum_{i=1}^{n}\left(-\frac{\sigma_\varepsilon^2}{(n-1)}\sum_{l=1}^{k-1}\frac{1}{\sigma_l^2}\left(x_i^l - \overline{X^l}\right)^2 + \sigma_\varepsilon^2 - \frac{\sigma_\varepsilon^2}{n}\right)$$

$$= -\sum_{l=1}^{k-1}\sigma_\varepsilon^2 + n\left(\sigma_\varepsilon^2 - \frac{\sigma_\varepsilon^2}{n}\right) = -(k-1)\sigma_\varepsilon^2 + n\sigma_\varepsilon^2 - \sigma_\varepsilon^2 = (n-k)\sigma_\varepsilon^2$$

$$\boldsymbol{\sigma_\varepsilon^2} = \frac{\mathbf{E}(\boldsymbol{RSS})}{(\boldsymbol{n}-\boldsymbol{k})}$$

Смещение оценок МНК

Пропущенная переменная

$Y = b_0 + b_1 X + b_2 Z + \varepsilon$ $\qquad$ $\mathrm{Cov}(X;Z) = \sigma_{XZ} \neq 0$

$$\sigma_{XY} = \mathrm{Cov}(X;Y) = \mathrm{Cov}(X; b_0 + b_1 X + b_2 Z + \varepsilon) = b_1 \mathrm{Cov}(X;X) + b_2 \mathrm{Cov}(X;Z)$$

$$\beta_1 = \frac{\sigma_{XY}}{\sigma_X^2} = b_1 + \boldsymbol{b_2 \frac{\sigma_{XZ}}{\sigma_X^2}}$$

$$\beta_0 = \overline{Y} - \beta_1 \overline{X} = b_0 + b_1 \overline{X} + b_2 \overline{Z} - \left(b_1 + b_2 \frac{\sigma_{XZ}}{\sigma_X^2}\right)\overline{X} = b_0 + \boldsymbol{b_2 \left(\overline{Z} - \frac{\sigma_{XZ}}{\sigma_X^2}\overline{X}\right)}$$

Система одновременных уравнений

$$\begin{cases} Y = b_0 + b_1 X + \varepsilon \\ X = Y + I \end{cases} \qquad \mathrm{Cov}(I;\varepsilon) = 0$$

$$\begin{cases} Y = b_0 + b_1 (Y + I) + \varepsilon \\ X = b_0 + b_1 X + \varepsilon + I \end{cases} \qquad \begin{cases} (1 - b_1) Y = b_0 + b_1 I + \varepsilon \\ (1 - b_1) X = b_0 + I + \varepsilon \end{cases}$$

$$\begin{cases} Y = \dfrac{b_0}{(1 - b_1)} + \dfrac{b_1}{(1 - b_1)} I + \dfrac{\varepsilon}{(1 - b_1)} \\ X = \dfrac{b_0}{(1 - b_1)} + \dfrac{1}{(1 - b_1)} I + \dfrac{\varepsilon}{(1 - b_1)} \end{cases}$$

$$\boldsymbol{\mathrm{Cov}(X;\varepsilon)} = \mathrm{Cov}\left(\frac{b_0}{(1 - b_1)} + \frac{1}{(1 - b_1)} I + \frac{\varepsilon}{(1 - b_1)}; \varepsilon\right) = \boldsymbol{\frac{\sigma_\varepsilon^2}{(1 - b_1)}}$$

$$\boldsymbol{\delta\beta_1} = \frac{\sigma_{X\varepsilon}}{\sigma_X^2} = \boldsymbol{\frac{1}{(1 - b_1)} \cdot \frac{\sigma_\varepsilon^2}{\sigma_X^2}}$$

$$\boldsymbol{\delta\beta_0} = \overline{\varepsilon} - \frac{\sigma_{X\varepsilon}}{\sigma_X^2}\overline{X} = \boldsymbol{-\frac{\overline{X}}{(1 - b_1)} \cdot \frac{\sigma_\varepsilon^2}{\sigma_X^2}}$$